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RADIATION PROJECT INTERNAL REPORT NUMBER 12

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INTRODUCTION

It has been recently shown by dePackh[1] that there exists an equilibrium relativistic flow in diodes which features a convergent sheath of electrons tending to flow parallel to the anode (parapotential flow) through which a core of current flows more or less directly to the anode (orthopotential flow). The specific impedance at any plane made up of electrostatic equipotentials is found to be proportional to the logarithm of the ratio of anode radius to the radius of the central current core. This suggests that if one replaces the central core of free electrons with a conductor and, furthermore, tapers it and the anode, one can achieve a match between a high source impedance and a low impedance tube. One is led to consider, therefore, coaxial cone geometries such as indicated in figure 1.

There still exists the question of whether this flow can ever be attained subsequent to the violent events characteristic of pulse generation in relativistic, high current diodes such as studied in the laboratory. It may well be that the entire beam life is a transient and devoid of such an equilibrium. As an aid in answering this question, we plan to study the problem in a time-dependent way on the computer. A key part of such a program is an expression for the force between macro-particles which simulate the beam of electrons in the presence of the electrodes...in this case, rings of charge flowing between coaxial cones. It is the purpose of this note to supply the necessary Green's function from which all interparticle forces, magnetic and electric, can be derived. Another report[2] will spell out the calculation of forces from the Green's function and detail all the necessary expansions necessary to make the formal expressions amenable to computer treatment.

NATURE OF HARMONIC SOLUTIONS IN CONAL REGIONS

The problem is best solved in spherical coordinates, r, θ, ϕ. The conal surfaces are then given by equations of the type

$$\theta = \text{constant}$$

if we put their vertices at the origin and their axes along the z-axis, as is shown in figure 2.

It is well known that the harmonic solutions in spherical coordinates can be written in the form

$$\Phi(r,\theta,\phi) = (A_n r^n + B_n r^{-n-1}) P_n^m(\cos\theta)\cos m\phi$$

Where P_n^m are Legendre functions. Since we will be considering the region between two cones, the two axes, $\cos\theta = \pm 1$, are excluded so that the degree of the Legendre function is not restricted to integer values.[3] We can determine the degree from the condition that $A_r{}^n + Br^{-n-1}$ be zero on the surfaces of two conducting spheres of radii $r = a$, $r = b$ which intersect the cones. We recover our problem by letting one radius go to infinity while the other shrinks to zero.

We have

$$Aa^n + Ba^{-n-1} = 0$$
$$Ab^n + Bb^{-n-1} = 0$$

which gives

$$(b/a)^{2n+1} = 1$$

or

$$(2n+1)\log(b/a) + 2m\pi i = 0$$

where m is any integer of either sign. Thus,

$$n = -\tfrac{1}{2} + m\pi i/\log \frac{b}{a} .$$

Now, if either a or b $\to \infty$ while the other goes to zero we are left with the following expression for the degree of the Legendre function

$$n = -\tfrac{1}{2} + i\tau$$

where τ varies continuously over positive and negative values. From the definition of the Legendre function in terms of the hypergeometric function[4]

$$P_n^m(\mu) = \frac{1}{\Pi(-m)} \left(\frac{\mu+1}{\mu-1} \right)^{m/2} . \; F(-n,\ n+1;\ 1-m;\ \frac{1-\mu}{2})$$

we see that

$$P_n^m(\mu) = P_{-n-1}^m(\mu)$$

which for our case means

$$P_{-\frac{1}{2}+i\tau}^m(\mu) = P_{-\frac{1}{2}-i\tau}^m(\mu) \quad \text{for } \mu = x,\ -1 < x < 1$$

Thus we can restrict our attention to

$$P_{-\frac{1}{2}+i\tau}^m(\mu) \qquad 0 \le \tau \le \infty.$$

These are termed <u>conal harmonics</u>.

INVERSE DISTANCE IN TERMS OF CONAL HARMONICS

We need to know the potential at r,θ,ϕ of a unit charge at r',θ',ϕ' in terms of the conal harmonics. The potential is conventionally written as

$$V(r,r',\gamma) \equiv 1/|\vec{r}-\vec{r}\,'| = 1/\sqrt{r^2+r'^2 - 2rr' \cos \gamma}$$

where

$$\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi-\phi') \; .$$

Now, let

$$r = e^{\sigma} \quad \text{and} \quad r' = e^{\sigma'} \quad .$$

Then

$$V(r,r',\gamma) \rightarrow V(\sigma-\sigma', \sigma+\sigma', \gamma) = e^{-\frac{1}{2}(\sigma+\sigma')} \frac{1}{\sqrt{2\cosh (\sigma-\sigma') - 2\cos \gamma}} \; .$$

Now we use Fourier's repeated integral theorem[5] to write

$$V(\sigma-\sigma', \sigma+\sigma',\gamma) = e^{-\frac{1}{2}(\sigma+\sigma')} \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \cos u(t-\sigma+\sigma') \, V(t,\sigma+\sigma'\gamma) \, du dt$$

$$= e^{-\frac{1}{2}(\sigma+\sigma')} \frac{2}{\pi} \int_0^{\infty} \cos u(\sigma-\sigma')du \int_0^{\infty} \frac{\cos ut \, dt}{\sqrt{2\cosh t-2\cos\gamma}}$$

where we have made use of the fact that $V(t, \sigma+\sigma', \gamma)$ is even in t, so that

$$\int_{-\infty}^{\infty} V(t, \sigma+\sigma', \gamma) \sin ut \, dt = 0.$$

From an integral expression for the Legendre function[6] for m=0 we have

$$P_\nu(z) = -\sqrt{\frac{2}{\pi}} \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \frac{\sin \nu\pi \cosh (\nu+\frac{1}{2})t \, dt}{\sqrt{2 \cosh t + 2z}} .$$

Let $z = \cos \theta$ and $\nu = -\frac{1}{2} + i\tau$ so that

$$P_{-\frac{1}{2}+i\tau}(\cos \theta) = \frac{2}{\pi} \cosh \tau\pi \int_0^\infty \frac{\cos \tau\phi \, d\phi}{\sqrt{2 \cos \theta + 2 \cosh \phi}}$$

and hence

$$V(\sigma-\sigma', \sigma+\sigma', \gamma) = e^{-\frac{1}{2}(\sigma+\sigma')} \int_0^\infty \frac{\cos u(\sigma-\sigma') \, P_{-\frac{1}{2}+iu}(-\cos \gamma)}{\cosh \pi u} du$$

which gives us $1/|\vec{r}-\vec{r}'|$ in terms of the conal harmonic. In order to express this formula in terms of θ, θ' and $\phi-\phi'$ we must use addition formulae for the conal functions[7]

$$P_{-\frac{1}{2}+i\tau}(-\cos \gamma) = P_{-\frac{1}{2}+\tau}(\cos \theta') \, P_{-\frac{1}{2}+i\tau}(-\cos \theta) +$$

$$+ 2 \sum_{m=1}^{\infty} \frac{(-)^m}{(\tau^2 + \frac{1}{4}) \cdots (\tau^2 + \frac{(2m-1)^2}{4})} P^m_{-\frac{1}{2}+i\tau}(\cos \theta') P_{-\frac{1}{2}+i\tau}(-\cos \theta)$$

$$\cos m(\phi-\phi') \quad \text{for } \theta > \theta'.$$

We remove consideration of the coordinate ϕ by integrating to get the <u>ring potential</u>

$$V(r,\theta;r',\theta')=\begin{cases} \dfrac{2\pi}{\sqrt{rr'}}\displaystyle\int_0^\infty \dfrac{\cos(\tau \log \frac{r}{r'})}{\cosh \pi\tau} P_{-\frac{1}{2}+i\tau}(\cos\theta)P_{-\frac{1}{2}+i\tau}(-\cos\theta')\,d\tau, & \theta' > \theta \\ \dfrac{2\pi}{\sqrt{rr'}}\displaystyle\int_0^\infty \dfrac{\cos(\tau \log \frac{r}{r'})}{\cosh \pi\tau} P_{-\frac{1}{2}+i\tau}(-\cos\theta)P_{-\frac{1}{2}+i\tau}(\cos\theta)\,d\tau, & \theta' < \theta \end{cases} .$$

To take account of the conal boundaries we write the potential as a sum

$$V(r,\theta;r',\theta') = \frac{1}{|\vec{r}-\vec{r}'|} + u_1 + u_2$$

where u_1 and u_2 remove the singularity at $\vec{r}=\vec{r}'$ and satisfy the zero potential conditions on the boundaries in the following way.

$$u_2(\theta_1) = 0 \qquad u_2(\theta_2) = -1/|\vec{r}(\theta_2) - \vec{r}'(\theta')|$$

$$u_1(\theta_2) = 0 \qquad u_1(\theta_1) = -1/|\vec{r}(\theta_1) - \vec{r}'(\theta')| .$$

One can immediately write down factors in u_1 and u_2 which satisfy these conditions and arrive at the following expressions which complete the formal solution.

Let $P_\tau \equiv P_{-\frac{1}{2}+i\tau}$, then

$$V(r,\theta;\, r',\theta') = \frac{2\pi}{\sqrt{rr'}} \int_0^\infty \frac{\cos(\tau \log \frac{r}{r'})}{\cosh \pi\tau} \Big\{ P_\tau(\cos\theta)\, P_\tau(-\cos\theta')$$

$$- \frac{P_\tau(\cos\theta) - P_\tau(\cos\theta_1)}{P_\tau(\cos\theta_2) - P_\tau(\cos\theta_1)}\, P_\tau(\cos\theta_2) \Big[P(-\cos\theta') - P(-\cos\theta_1) \Big]$$

$$- \frac{P_\tau(-\cos\theta') - P(-\cos\theta_2)}{P_\tau(-\cos\theta_1) - P(-\cos\theta_2)}\, P_\tau(-\cos\theta_1) \Big[P_\tau(\cos\theta) - P_\tau(\cos\theta_2) \Big]$$

$$- P_\tau(\cos\theta_2)\, P_\tau(-\cos\theta_1) \Big\}\, d\tau \qquad \text{for } \theta < \theta'.$$

For $\theta > \theta'$, let $\theta \leftrightarrow \theta'$ in the above formula. Note that the Green's function has the requisite symmetry under the reflection $\theta \leftrightarrow \theta'$, $r \leftrightarrow r'$, and satisfies the condition of zero potential when either source or field point is on the boundary.

The last integration over degree, τ, must be accomplished by a preliminary computer run.

REFERENCES

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6. A. Erdelyi (ed.), <u>Higher Transcendental Functions</u>, Vol. I, Bateman Manuscript Project, McGraw-Hill, New York, 1953, pg. 156.

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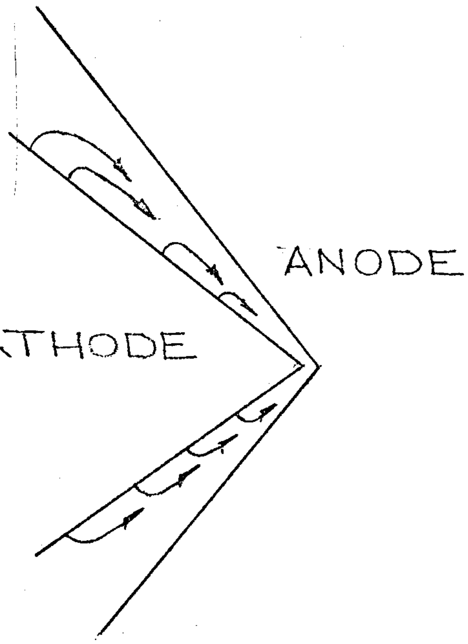

Fig. 1

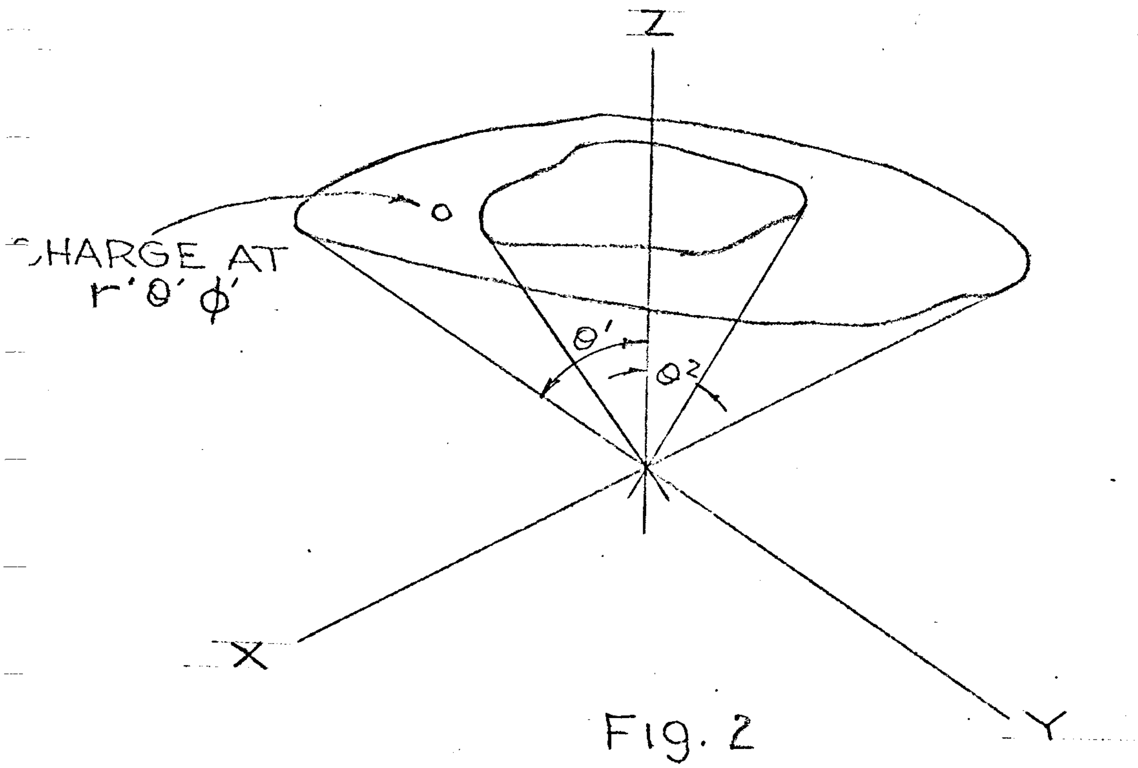

Fig. 2